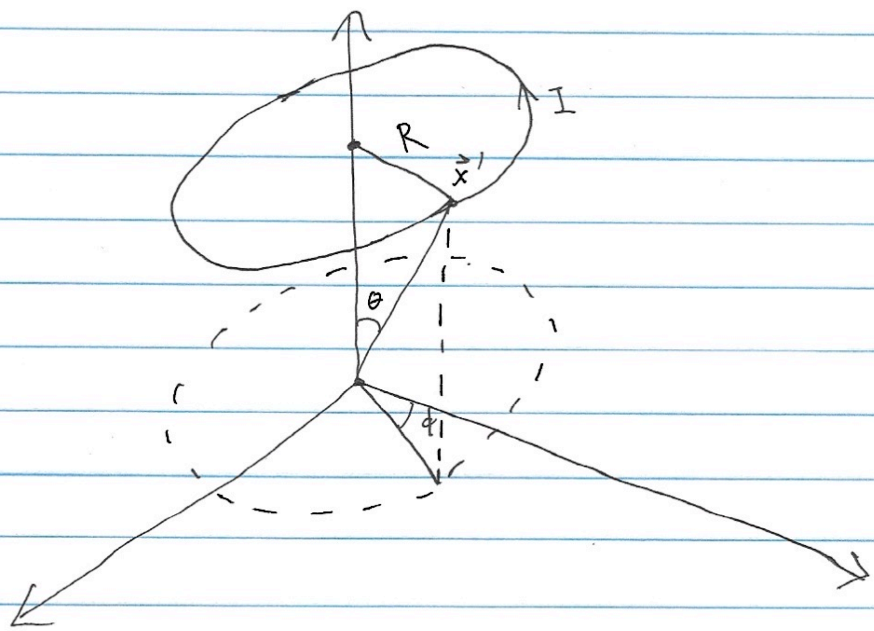
Jackson 5.3

The formula for $\vec{B}$ is given by

$$d\vec{B} = \frac{\mu_0 I}{4\pi} d\vec{l}' \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}.$$

First consider a single circular circuit with current I running counterclockwise, centered at $\hat{z}$, with radius R, we wish to find the field produced, with P put at origin, so $\vec{x} = (0, 0, 0)$



$$\vec{x}' = \frac{R}{\tan\theta} \hat{z} + R [\cos\phi \, \hat{x} + \sin\phi \, \hat{y}]$$

$$\vec{x} - \vec{x}' = -R [\cot\theta \, \hat{z} + \cos\phi \, \hat{x} + \sin\phi \, \hat{y}]$$

$$|\vec{x} - \vec{x}'|^3 = \left(\frac{R}{\sin\theta}\right)^3$$

$$\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \frac{-\sin^3\theta}{R^2}\left[\cot\theta\,\hat{z} + \cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right].$$

$$d\vec{l}' = d|l|\,\hat{l}' = R\left[\cos\phi\,\hat{y} - \sin\phi\,\hat{x}\right]d\phi$$

$$\vec{B} = kI\int d\vec{l}' \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$= kI\int \frac{-\sin^3\theta}{R}\left[\cos\phi\,\hat{y} - \sin\phi\,\hat{x}\right] \times \left[\cot\theta\,\hat{z} + \cos\phi\,\hat{x} + \sin\phi\,\hat{y}\right] d\phi$$

$$= kI\int -\frac{\sin^3\theta}{R}\left\{\begin{array}{l}\left[(-\sin\phi)\sin\phi - \cos\phi\cos\phi\right]\hat{z} \\ + \cos\phi\,\frac{\cos\theta}{\sin\theta}\,\hat{x} \\ + \sin\phi\,\frac{\cos\theta}{\sin\theta}\,\hat{y}\end{array}\right\} d\phi$$

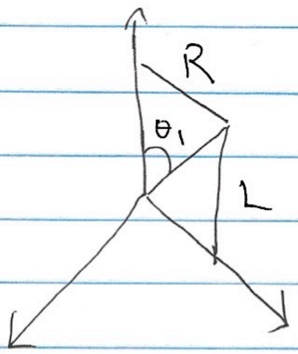
Evidently, $\int_0^{2\pi}\cos\phi\,d\phi$ and $\int_0^{2\pi}\sin\phi\,d\phi$ vanish so x, y components vanish and we are left with

$$\vec{B} = kI\int \frac{\sin^3\theta}{R}\,\hat{z}\,d\phi$$

$$= \frac{\mu_0 I}{4\pi}(2\pi)\frac{\sin^3\theta}{R}\,\hat{z}$$

$$* \quad = \frac{\mu_0 I}{2R}\sin^3\theta\,\hat{z}$$

We have found the field produced by a single current loop. To find the field produced by the solenoid, we need to integrate the "loop density" over the length of the solenoid.



$$L = R \cot\theta = R \frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow dL = -\frac{R}{\sin^2\theta} \, d\theta$$

$$d(\#\text{ loops}) = \frac{d(\#\text{ loops})}{dL} \, dL$$

$$= N \, dL$$

$$* \quad = N \left[-\frac{R}{\sin^2\theta} \right] d\theta$$

$$\Rightarrow \vec{B}_{\text{solenoid}} = \left[\int_{\theta_1}^{\pi/2} + \int_{\pi/2}^{\pi - \theta_2} \right] \left[\vec{B}_{\text{single loop}} \, d(\#\text{ loops}) \right]$$

$$= \left[\int_{\theta_1}^{\pi/2} + \int_{\pi/2}^{\pi - \theta_2} \right] \left[\frac{\mu_0 I}{2R} \sin^3\theta \, \hat{z} \, N \left[\frac{-R}{\sin^2\theta} \right] d\theta \right]$$

$$= \left[\int_{\theta_1}^{\pi/2} + \int_{\pi/2}^{\pi - \theta_2} \right] \frac{\mu_0 I N}{2} (-\sin\theta) \, \hat{z} \, d\theta$$

$$= \frac{\mu_0 N I}{2} \left[\int_{\pi/2}^{\theta_1} + \int_{\pi-\theta_2}^{\pi/2} \right] \sin\theta \, d\theta \; \hat{z}$$

$$= \frac{\mu_0 N I}{2} \int_{\pi-\theta_2}^{\theta_1} \sin\theta \, d\theta \; \hat{z}$$

$$= \frac{\mu_0 N I}{2} \left[-\cos\theta_1 + \cos(\pi-\theta_2) \right] \hat{z}$$

$$= \frac{\mu_0 N I}{2} \left[\cos\theta_1 + \cos\theta_2 \right] (-\hat{z})$$

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